Extremely amenable automorphism groups

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- The relation between the (EA) of $\mathrm{Aut}(\mathcal{M})$ and the (approximate) Ramsey properties of $\mathrm{Age}(\mathcal{M})$ (the KPT-correspondence).

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- The "metric" theory for the case of Banach spaces.
- The Gurarij space and the $L_p[0, 1]$-spaces.

Part I: Basics

1. Topological Dynamics
   - Extreme Amenability, Universal Minimal Flows
   - UMF vs EA; how to prove EA
2. (Metric) Fraïssé Theory
   - First order structures
   - KPT correspondence; Structural Ramsey Properties
   - Structural Ramsey Theorems
   - Metric structures

Part II: An example of metric structures: Banach spaces

Part III: Three Examples

6. Gurarij space
   - $\{\ell_\infty^n\}_n$ have have the (ARP)
   - The ARP of Finite dimensional Normed spaces
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7. L_p-spaces
   - L_p (sometimes) is a Fraïssé space
   - $\{\ell_p^n\}$ have the (ARP)

Extreme Amenability

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Definition

A topological group G is called extremely amenable (EA) when every continuous action (flow) $G \curvearrowright K$ on a compact K has a fixed point; that is, there is $p \in K$ such that $g \cdot p = p$ for all $g \in G$.

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EA groups are amenable (G is amenable iff every affine flow $G \curvearrowright K$ on a compact convex space K has a fixed point).

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is ... $\phi(x)$.

We consider the commutative C^*-algebra of right uniformly continuous and bounded $f : G \to \mathbb{C}$, and represent it as $C(S(G))$ (Gelfand); any minimal flow of $S(G)$ is G-isomorphic to $\mathcal{M}(G)$.

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Question

Compute universal minimal flows.

Examples of EA groups

1. The unitary group $\mathbb{U}$ of linear isometries of the separable infinite dimensional Hilbert space $\mathbb{H}$, endowed with its strong operator topology SOT (i.e. the pointwise convergence topology) (Gromov-Milman);

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3. The group of isometries of the Urysohn space with its pw. conv. top. (Pestov);

4. The group of linear isometries of the Lebesgue spaces $L_p[0, 1]$, $1 \leq p \neq 2 < \infty$, with the SOT (Giordano-Pestov);

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6. The Automorphism group of the ordered universal $\mathbb{F}$-vector space $\mathbb{F}^{<\infty}$, $\mathbb{F}$ finite field, is extremely amenable (K-P-T);

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7. The group of linear isometries of the Gurarij space $\mathbb{G}$ (Bartosova-LA-Lupini-Mbombo).

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4. $M(\mathrm{Aut}(\mathbb{P})) = \mathbb{P}$, where $\mathbb{P}$ is the Poulsen simplex, the unique compact metrizable Choquet simplex whose extreme points are dense (B-LA-L-M).

UMF and EA

Proposition (Ben Yaacov-Melleray-Tsankov)

Suppose that G is a polish group (i.e. separable and complete metrizable topological group). If the umf $M(G)$ is metrizable, then there is an EA subgroup H of G such that $M(G)$ is the completion of G/H.

Proving that a group is EA

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While the first seems a restricted approach, the second is general, as proved by Melleray.

Aut(X) is extremely amenable

X	**Method**
$\mathbb{H}$	Lévy
$\mathbb{Q}$	KPT
$\mathbb{U}$	Lévy and KPT
$L_p[0, 1]$	Lévy and KPT
$\mathbb{B}$	KPT
$\mathbb{F}^{<\infty}$	KPT
$\mathbb{G}$	KPT

Table: Methods to prove extreme amenability

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Definition (Ultrahomogeneity)

A first order structure $\mathcal{M}$ is called ultrahomogeneous when for every finitely generated substructure $\mathcal{N}$ of $\mathcal{M}$ and every embedding $\phi : \mathcal{N} \to \mathcal{M}$ there is an automorphism $g \in \mathrm{Aut}(\mathcal{M})$ such that $g \restriction N = \phi$.

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Fraïssé theory tells that countable ultrahomogeneous structures are the Fraïssé limits of Fraïssé classes (hereditary property, joint embedding property, and amalgamation property).

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Proposition (Representation Theorem I)

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Permutations of $\mathbb{N}$ with the topology of point-wise convergence.

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Proof.

For suppose that G is a closed subgroup of $\mathcal{S}_\infty$; For each $k \in \mathbb{N}$, consider the canonical action $G \curvearrowright \mathbb{N}^k$, $g \cdot (a_j)_{j<k} := (g(a_j))_{j<k}$, and let $\{O_j^{(k)}\}_{j \in I_k}$ be the enumeration of the corresponding orbits. Let $\mathcal{L}$ be the relational language, $\{R_j^{(k)} \, : \, k \in \mathbb{N}, \, j \in I_k\}$, each $R_j^{(k)}$ being a k-ari relational symbol. Now $\mathbb{N}$ is an $\mathcal{R}$-structure $\mathcal{M}$ naturally,

$$(R_j^{(k)})^{\mathcal{M}} := O_j^{(k)}.$$

It is easy to see that $\mathcal{M}$ is ultrahomogeneous, and that $G \subseteq \mathrm{Aut}(\mathcal{M})$ is dense in G, so, equal to G.

□

Kechris-Pestov-Todorcevic correspondence

Given two first order structures of the same sort $\mathbf{A}, \mathbf{B}$, let $\mathrm{emb}(\mathbf{A}, \mathbf{B})$ be the collection of all 1-1 morphisms $h : \mathbf{A} \to \mathbf{B}$.

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Definition (Structural Ramsey Property)

Let $\mathcal{F}$ be a class of finitely generated first order structures of the same sort. The class $\mathcal{F}$ has the **Structural** Ramsey Property (RP) if for every $\mathbf{A}, \mathbf{B} \in \mathcal{F}$ and every $r \in \mathbb{N}$ there is $\mathbf{C} \in \mathcal{F}$ such that for every coloring $c : \mathrm{emb}(\mathbf{A}, \mathbf{C}) \to r$ there is $\varrho \in \mathrm{emb}(\mathbf{B}, \mathbf{C})$ such that $\varrho \circ \mathrm{emb}(\mathbf{A}, \mathbf{B})$ is c-monochromatic.

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Theorem (Kechris-Pestov-Todorcevic)

Let M be a countable ultrahomogeneous structure. TFAE:

1. $\mathrm{Aut}(M)$ *is extremely amenable;*
2. $\mathrm{Age}(M)$ *has the Ramsey property (RP).*

The Classical Ramsey Theorem

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Proposition (F. P. Ramsey)

For every $k, m, r \in \mathbb{N}$ there is $n \geq k$ such that every r-coloring

$$c : [n]^k \to r$$

has a monochromatic set of the form $[A]^k$ for some $A \subseteq n$ of cardinality m.

This is equivalent to the following: Let $\mathrm{emb}(k,n)$ be the collection of all injections $f : k \to n$ (so, no structure).

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1. *The class of finite linear orderings has the Ramsey property, and*
2. $\mathrm{Aut}(\mathbb{Q}, <)$ *is extremely amenable.*

The Dual Ramsey Theorem (DRT)

Let $\mathcal{E}_n^d$ be the set of all partitions of n into d-many pieces. Given a partition $\mathcal{Q} \in \mathcal{E}_n^m$, and $d \leq m$, let $\langle \mathcal{Q} \rangle^d$ be set of all partitions $\mathcal{P} \in \mathcal{E}_n^d$ *coarser* than $\mathcal{Q}$.

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Theorem (Dual Ramsey by Graham and Rothschild)

For every d, m and r there exists n such that for every coloring $c : \mathcal{E}_n^d \to r$ there exists $\mathcal{Q} \in \mathcal{E}_n^m$ such that $c \restriction \langle \mathcal{Q} \rangle^d$ is constant.

By a simple dual argument, this is equivalent to the following. Given $k, n \in \mathbb{N}$, we consider $\mathcal{P}(k)$ and $\mathcal{P}(n)$ as boolean algebras, and then let $\mathrm{emb}(k, n)$ be the collection of all ordered boolean embeddings $f : \mathcal{P}(k) \to \mathcal{P}(n)$, i.e., such that $\min f(\{i\} < \min f(\{j\})$ for every $i < j < k$. The dual Ramsey theorem can be restated as follows.

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Theorem (DR, Boolean version)

For every k, m and r in $\mathbb{N}$ there is some $n \in \mathbb{N}$ such that every r-coloring $c : \mathrm{emb}(\mathcal{P}(k), \mathcal{P}(n)) \to r$ has a monochromatic set of the form $\varrho \circ \mathrm{emb}(\mathcal{P}(k), \mathcal{P}(m))$ for some $\varrho \in \mathrm{emb}(\mathcal{P}(m), \mathcal{P}(n))$; consequently,

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1. *The class of finite, canonically ordered, boolean algebras has the Ramsey property, and*
2. *The automorphism group of the canonically ordered countable atomless boolean algebra is extremely amenable.*

Metric structures

The rest of the examples are also groups of algebraic automorphisms that are in addition isometries. First order structures are the discrete version of metric structures $\mathcal{M} = (M, (F^{\mathcal{M}})_{F\in\mathcal{F}}, (R^{\mathcal{M}})_{R\in\mathcal{F}})$: Roughly speaking:

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1. d is a bounded and complete metric on M;
2. n-ari function symbols F are interpreted as uniformly continuous functions $F^{\mathcal{M}} : M^n \to M$;

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The rest of the examples are also groups of algebraic automorphisms that are in addition isometries. First order structures are the discrete version of metric structures $\mathcal{M} = (M, (F^{\mathcal{M}})_{F\in\mathcal{F}}, (R^{\mathcal{M}})_{R\in\mathcal{F}})$: Roughly speaking:

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4. operator spaces, etc.

Approximate Ultrahomogeneity

Definition (Approximate Ultrahomogeneity)

A metric structure $\mathcal{M}$ is called approximate ultrahomogeneous when for every finitely generated substructure $\mathcal{N}$ of $\mathcal{M}$ and every embedding $\phi : \mathcal{N} \to \mathcal{M}$ there is an automorphism $g \in \mathrm{Aut}(\mathcal{M})$ such that $\widehat{d}(g \upharpoonright N, \phi) < \varepsilon$.

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Metric Fraïssé theory tells that countable ultrahomogeneous structures are the Fraïssé limits of Fraïssé classes (hereditary property, joint embedding property, and near amalgamation property).

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Proposition (Representation Theorem II; Melleray)

Every polish group G is the automorphism group of an approximate ultrahomogeneous metric structure.

Metric KPT correspondence

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Ramsey theorem, Dual Ramsey Theorem, Graham-Leeb-Rothschild, Nešetřil...

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$\mathbb{Q}$ as order, $\mathbb{B}_{<\infty}$, $\mathbb{F}^{<\infty}$, $\mathbb{U}$

An example of metric structures: Banach spaces

Outline

Fraïssé Banach spaces

Definition

Let E be an infinite dimensional Banach space, and let $\mathcal{G} \preceq \mathrm{Age}(E)$.

Fraïssé Banach spaces

Age(E):=Finite dimensional subspaces of E.
$\mathcal{F} \preceq \mathcal{G}$ when for every $X \in \mathcal{F}$ there is $Y \in \mathcal{G}$ isometric to X.

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Let E be an infinite dimensional Banach space, and let $\mathcal{G} \preceq \mathrm{Age}(E)$.

- E is *$\mathcal{G}$-homogeneous* ($\mathcal{G}-$H) when for every $X \in \mathcal{G}$ and every and every $\gamma, \eta \in \mathrm{Emb}(X, E)$ there is some $g \in \mathrm{Iso}(E)$ such that $g \circ \gamma = \eta$; in other words, when for each $X \in \mathcal{G}$, the natural action $\mathrm{Iso}(E) \curvearrowright \mathrm{Emb}(X, E)$ by composition is transitive.

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Linear $\gamma : X \to E$ with $\|Tx\| = \|x\|$

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$g \cdot \gamma := g \circ \gamma$

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- E is is called *approximately $\mathcal{G}$-homogeneous* (A$\mathcal{G}$H) when for every $X \in \mathcal{G}$ and every $\varepsilon > 0$ the natural action by composition $\mathrm{Iso}(E) \curvearrowright \mathrm{Emb}(X, E)$ is ε-transitive, that is, whenever $\gamma, \eta \in \mathrm{Emb}(X, E)$ there is $g \in \mathrm{Iso}(E)$ such that $\|g \circ \gamma - \eta\| < \varepsilon$.

Fraïssé Banach spaces

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- E is is called *weak $\mathcal{G}$-Fraïssé* when for every $X \in \mathcal{G}$ and every $\varepsilon > 0$ there is $\delta \geq 0$ such that $\mathrm{Iso}(E) \curvearrowright \mathrm{Emb}_\delta(X, E)$ is ε-*transitive*.

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  $\mathcal{G}_k =$ spaces in $\mathcal{G}$ of dimension k.

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When $\mathcal{G} = \mathrm{Age}(E)$, then we will use *ultrahomogeneus* (uH), *approximately ultrahomogeneous* (AuH$^+$), weak Fraïssé and Fraïssé for the corresponding $\mathcal{G}$-homogeneities.

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- For every $1 \leq p < \infty$ the space $L_p[0,1]$ is $\{\ell_p^n\}_n$-Fraïssé . In fact, $L_p[0,1]$ is the *Fraïssé limit* of $\{\ell_p^n\}_n$.

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- Assume $p \in 2\mathbb{N}$, $p \geq 4$. For any $C \geq 1$ and $\delta \geq 0$, there are isometric $E, F \in \mathrm{Age}(L_p(0,1))$ such that for any bounded linear mapping $T : L_p(0,1) \to L_p(0,1)$, if $T \upharpoonright E \in \mathrm{Emb}_\delta(E,F)$, then $\|T\| \geq C$.

E-Kadets

Recall the gap or opening metric on $\mathrm{Age}_n(E)$ is defined by

$$\Lambda_E(X,Y) := \max\left\{\max_{x\in B_X}\min_{y\in B_Y}\|x-y\|_E, \max_{y\in B_Y}\min_{x\in B_X}\|x-y\|_E\right\};$$

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This induces the following *Gromov-Hausdorff* function, E-Kadets on $\mathrm{Age}_n(E)^2$, defined as

$$\gamma_E(X,Y) := \inf\{\Lambda_E(X_0,Y_0) \,:\, X_0,Y_0 \in \mathrm{Age}_n(E),\, X_0 \equiv X,\, Y_0 \equiv Y\}.$$

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Proof.

Wlog, we assume that $\mathcal{G} \subseteq \mathrm{Age}(E)$. Then,
$\gamma_E(X, Y) = \inf_{g \in \mathrm{Iso}(E)} \Lambda_E(gX, Y)$ □

Banach-Mazur

The *Banach-Mazur* pseudometric on $\mathrm{Age}_n(E)$:

$$d_{\mathrm{BM}}(X,Y) := \log(\inf_{T:X\to Y} \|T\| \cdot \|T^{-1}\|)$$

where the infimum runs over all isomorphisms $T : X \to Y$. It is well-known that d_{BM} defines a pre-compact topology on $\mathrm{Age}_n(E)$; that is, every sequence in $\mathrm{Age}_n(E)$ has a d_{BM}-convergent subsequence, not necessarily to an element of $\mathrm{Age}_n(E)$.

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It follows from this that the Hilbert and the Gurarij spaces are very special Fraïssé spaces: Recall that a Banach space Y is *finitely representable* in X if $\mathrm{Age}_k(Y)$ is included in the d_{BM}-closure $\overline{\mathrm{Age}_k(X)}^{\mathrm{BM}}$ of $\mathrm{Age}_k(X)$ for every k.

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Consequently,

3. *ℓ_2 is the minimal separable Fraïssé Banach space.*

$\mathcal{G}$-Fraïssé spaces are locally determined

Definition

Given a family $\mathcal{G}$ of finite dimensional spaces, let $[\mathcal{G}]$ be the class of all separable Banach spaces X such that there is an $\subseteq$-increasing sequence $(X_n)_n$ in $\mathcal{G}_X$ whose union is dense in X.

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it follows that $\mathcal{G}$ has the Joint embedding property: For every $X, Y \in \mathcal{G}$ there is $Z \in \mathcal{G}$ such that $\mathrm{Emb}(X, Z), \mathrm{Emb}(Y, Z) \neq \emptyset$.

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- $\mathcal{G}$ is a Fraïssé class when it is hereditary amalgamation class.

Fraïssé correspondence

Theorem

Suppose that $\mathcal{G}$ *is an amalgamation class. Then there is a unique separable* $\mathcal{G}$*-Fraïssé Banach space* E *such that* $E \in [\mathcal{G}]$*, called the* Fraïssé limit of $\mathcal{G}$ *and denoted by* $\operatorname{Flim}\mathcal{G}$.

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Being Fraïssé is an ultra property. Recall that given a Banach space E, and given a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$,

- we write $E_{\mathcal{U}}$ to denote the *ultrapower* $E^{\mathbb{N}}/\mathcal{U}$.

Fraïssé and ultrapowers

Being Fraïssé is an ultra property. Recall that given a Banach space E, and given a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$,

- we write $E_{\mathcal{U}}$ to denote the *ultrapower* $E^{\mathbb{N}}/\mathcal{U}$.
- We denote by $\mathrm{Iso}(E)_{\mathcal{U}}$ the subgroup of $\mathrm{Iso}(E_{\mathcal{U}})$ consisting of all isometries of the ultrapower $E_{\mathcal{U}}$ of the form $[(x_n)_n]_{\mathcal{U}} \mapsto [(g_n(x_n))_n]_{\mathcal{U}}$ for some sequence $(g_n)_n \in \mathrm{Iso}(E)^{\mathbb{N}}$.

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It is well known that $\mathrm{Age}(E_{\mathcal{U}}) \equiv \overline{\mathrm{Age}(E)}^{\mathrm{BM}}$.

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In particular, it follows that when E is Fraïssé, its ultrapowers is Fraïssé and ultrahomogeneous.

ARP for finite dimensional normed spaces

Given two Banach spaces X and Y, and $\delta \geq 0$, let $\mathrm{Emb}_\delta(X, Y)$ be the collection of all linear 1-1 bounded functions $T : X \to Y$ such that $\|T\|, \|T^{-1}\| \leq 1 + \delta$.

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Definition

A collection $\mathcal{F}$ of finite dimensional normed spaces has the Approximate Ramsey Property (ARP) when for every $F, G \in \mathcal{F}$ and $\varepsilon > 0$ there exists $H \in \mathcal{F}$ such that every continuous coloring c of $\mathrm{Emb}(F, H)$ ε-stabilizes in $\varrho \circ \mathrm{Emb}(F, G)$ for some $\varrho \in \mathrm{Emb}(G, H)$, that is,

$$\mathrm{osc}(c \restriction \varrho \circ \mathrm{Emb}(F, G)) < \varepsilon.$$

Comparing different Ramsey Propeties

Definition

A collection $\mathcal{F}$ of finite dimensional normed spaces has the Discrete (ARP) when for every $F, G \in \mathcal{F}$, $\varepsilon > 0$ and $r \in \mathbb{N}$ there exists $H \in \mathcal{F}$ such that every coloring c of $\mathrm{Emb}(F, H) \to r$ has an ε-monochromatic set of the form $\varrho \circ \mathrm{Emb}(F, G)$ for some $\varrho \in \mathrm{Emb}(G, H)$.

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$\mathcal{F}$ has the (ARP) *if and only if $\mathcal{F}$ has the discrete* (ARP).

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Step I

Proposition

Let G be a topological group, $\mathrm{Iso}(E) \curvearrowright K$*, and suppose that* $\mathrm{Iso}(E) \cdot p$ *is dense K. The following are equivalent.*

- i *there is a fixed point for the action* $\mathrm{Iso}(E) \curvearrowright K$*.*
- ii *For every entourage U in K and every finite set* $F \subseteq \mathrm{Iso}(E)$ *there is some* $g \in \mathrm{Iso}(E)$ *such that* $Fg \cdot p$ *is U-small, that is for every* $f_0, f_1 \in F$ *one has that* $(f_0 g \cdot p, f_1 g \cdot p) \in U$*.*

Proof.

i implies ii For suppose that $q \in K$ is a fixed point; Fix $F \subseteq G$ finite and an entourage U; let V be an entourage such that $V \circ V \subseteq U$. Using that $g\cdot : K \to K$ is uniformly continuous, we find an entourage W such that $gW \subseteq V$ for every $g \in F$. Let $h \in G$ be such that $(h \cdot p, q) \in W$. It follows that $(gh \cdot p, q) = (gh \cdot p, gq) \in V$ for all $g \in F$; hence $(gh \cdot p, g'h \cdot p) \in U$.

□

Proof.

ii implies i For every finite set F and entourage U choose $g_{F,U} \in G$ such that $(F \cup \{e\}) \cdot g_{F,U}p$ is U-small, hence $fg_F p \in U[g_{F,U}p]$ for every F and U. Then any accumulation point q of $\{g_{F,U}\}_{F,U}$ is a fixed point.

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1. Fix $\mathrm{Iso}(E) \curvearrowright K$, $p \in K$, an entourage U and a finite set $F \subseteq \mathrm{Iso}(E)$. Set $H := F^{-1}$.

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4. For each n, let d_n be the pseudometric on Iso(E), $d_n(g,h) := \|g \restriction X_n - h \restriction X_n\|$.

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5. Since the sequence of pseudometrics $(d_n)_n$ defines the SOT on $\mathrm{Iso}(E)$ and since $G \to K$, $g \mapsto g^{-1}p$ is uniformly continuous there is some $n \in \mathbb{N}$ and $\delta > 0$ such that $d_n(g,h) \leq \delta$ implies that $(g^{-1} \cdot p, h^{-1} \cdot p) \in V$.

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9. We define the coloring $c : \mathrm{Emb}(X_n, Z) \to r$ for $\gamma \in \mathrm{Emb}(X_n, Z) \to r$ by choosing $g \in \mathrm{Iso}(E)$ such that $\|g \restriction Y - \gamma\| \le \delta/3$, and then by declaring $c(\gamma) = j$ if j is (the first) such that $g^{-1}p \in V[x_j]$.

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11. Choose $h \in \mathrm{Iso}(E)$ such that $\|h \upharpoonright Y - \varrho\| \le \delta/3$.
12. Then, for every $f \in H$, setting $\eta := f \upharpoonright X_n$, then $d_n(h \circ f, g_\eta) \le \delta$, and $g_\eta^{-1} \cdot p \in V[x_j]$.

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12. Then, for every $f \in H$, setting $\eta := f \upharpoonright X_n$, then $d_n(h \circ f, g_\eta) \leq \delta$, and $g_\eta^{-1} \cdot p \in V[x_j]$.
13. Consequently, $(f_0 \circ h^{-1} \cdot p, f_1 \circ h^{-1} \cdot p) \in U$ for every $f_0, f_1 \in F$, as desired.

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- Given two metric spaces (A, d_A) and (B, d_B), let $\mathrm{Lip}(A, B)$ be the collection of 1-Lipschitz mappings from A to B.
- When A is compact, we endow it with the uniform metric $d(c, d) := \sup_{a \in A} d_B(c(a), d(a))$. Observe that when B is also compact, $(\mathrm{Lip}(A, B), d)$ is also compact.

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- For each $W \in \mathrm{Age}(E)$, let $\langle W \rangle := \{X \in \mathrm{Age}(E) \, : \, W \subseteq X\}$. Note that $\{\langle W \rangle\}_{W \in \mathrm{Age}(E)}$ has the finite intersection property. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathrm{Age}(E)$ containing all $\langle W \rangle$.

- Define the ultraproduct $\mathrm{Lip}_{\mathcal{U}}(\mathrm{Emb}(X,E),[0,1]) := (\prod_{X\subseteq Y\in \mathrm{Age}(E)} \mathrm{Lip}(\mathrm{Emb}(X,Y),[0,1])/\sim_{\mathcal{U}}$, where $(c_Y)_Y \sim_{\mathcal{U}} (d_Y)_Y$ if and only if for every $\gamma_0,\dots,\gamma_{n-1}\in \mathrm{Emb}(X,E)$, and every $\varepsilon>0$, $\{Y\in \langle \sum_{j<n} \mathrm{Im}\gamma_j\rangle \, : \, |\max_{j<n}|c_Y(\gamma_j)-d_Y(\gamma_j)|\le \varepsilon\}\in \mathcal{U}$.

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- We consider the canonical action $\mathrm{Iso}(E)\curvearrowright \mathrm{Lip}(\mathrm{Emb}(X,E),[0,1]$, $(g\cdot c)(\gamma):=c(g\circ\gamma)$, and the (algebraic) action $\mathrm{Iso}(E)\curvearrowright \mathrm{Lip}_{\mathcal{U}}(\mathrm{Emb}(X,E),[0,1])$, $g\cdot[(c_Y)_Y]_{\mathcal{U}}=[(d_Y)_Y]_{\mathcal{U}}$, where $d_Y(\gamma):=c_{g(Y)}(g\circ\gamma)$.

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- We consider the canonical action $\mathrm{Iso}(E)\curvearrowright \mathrm{Lip}(\mathrm{Emb}(X,E),[0,1]$, $(g\cdot c)(\gamma):=c(g\circ\gamma)$, and the (algebraic) action $\mathrm{Iso}(E)\curvearrowright \mathrm{Lip}_{\mathcal{U}}(\mathrm{Emb}(X,E),[0,1])$, $g\cdot[(c_Y)_Y]_{\mathcal{U}}=[(d_Y)_Y]_{\mathcal{U}}$, where $d_Y(\gamma):=c_{g(Y)}(g\circ\gamma)$.
- Define $\Phi:\mathrm{Lip}(\mathrm{Emb}(X,E),[0,1]\to \mathrm{Lip}_{\mathcal{U}}(\mathrm{Emb}(X,E),[0,1])$, $\Phi(c)=(c_Y)_Y$, where $c_Y(\gamma):=c(\gamma)$.

Proposition

Φ *is a* $\mathrm{Iso}(E)$*-bijection.*

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Proof.

Suppose that $\Phi(c) = [(c_Y)_Y]_{\mathcal{U}}$ and $\Phi(g \cdot c) = [(d_Y)_Y]_{\mathcal{U}}$. Then for each Y and $\gamma \in \mathrm{Emb}(X, Y)$, $c_Y(\gamma) = c(\gamma)$ and $d_Y(\gamma) = (g \cdot c)(\gamma) = c(g \circ \gamma)$, so $g \cdot [(c_Y)_Y]_{\mathcal{U}} = [(d_Y)_Y]_{\mathcal{U}}$. It is easy to see that Φ is 1-1.

□

Proposition

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Proof.

Φ is onto: Suppose now that $\Phi(c) = \Phi(d)$. Let $[(c_Y)_Y]_{\mathcal{U}}$, and let $\gamma \in \mathrm{Emb}(X, E)$. Then the numerical sequence $(c_Y(\gamma))_{\mathcal{U}}$ is bounded, so the $\mathcal{U}$-limit $c(\gamma) := \lim_{Y \to \mathcal{U}} c_Y(\gamma)$ exists. It is ease to see that $c \in \mathrm{Lip}(\mathrm{Emb}(X, E), [0, 1])$ and that $\Phi(c) = [(c_Y)_Y]_{\mathcal{U}}$.

□

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4. Since Φ is a $\mathrm{Iso}(E)$-bijection, this is equivalent to prove that given $c \in \mathrm{Emb}(X, E) \to [0, 1]$ there is some $g \in \mathrm{Iso}(E)$ such that $\mathrm{Osc}(c \restriction g \circ \mathrm{Emb}(X, Y)) \leq \varepsilon$.

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5. It follows that for $\gamma,\eta \in \mathrm{Emb}(X,Y)$, $|d(\gamma) - d(\eta)| = |d(h\circ\gamma) - d(\eta)| \le \varepsilon/3$, and $|c(g\circ\gamma) - c(g\circ\eta)| \le 2\varepsilon/3 + |d(\gamma) - d(\eta)| \le \varepsilon$.

Résumé

- Fraïssé spaces are those spaces for which $\mathrm{Iso}(E) \curvearrowright \mathrm{Emb}_\delta(X, E)$ ε-transitively, and the dependance

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- Concerning ultrapowers, E is Fraïssé if and only if the subgroup $(\mathrm{Iso}(E))_{\mathcal{U}}$ of $\mathrm{Iso}(E_{\mathcal{U}})$ acts transitively on each $\mathrm{Emb}(X, E_{\mathcal{U}})$ for every separable (possibly infinite dimensional $X \subseteq E_{\mathcal{U}}$).

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- There is a local (KPT) and a global (KPT) that connects the extreme amenability of $\mathrm{Iso}(E)$ with the (ARP) of $\mathcal{G}$ or of $\mathrm{Age}(E)$, when $\mathcal{G}$ is an amalgamation class and E

Three examples

Outline

6. Gurarij space
   - $\{\ell_\infty^n\}_n$ have have the (ARP)
   - The ARP of Finite dimensional Normed spaces
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7. L_p-spaces
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First application of metric KPT correspondence

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There are also noncommutative analogues. ****** **falta** **** mention M-spaces.

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